JOURNAL OF APPROXIMATION THEORY 39, 320–323 (1983)

A Connected Separable Metric Space with a Dispersed Chebyshev Set

SUSANNA PAPADOPOULOU*

Katholische Universität Eichstätt and Mathematisches Institut der Universität Erlangen-Nürnberg

Communicated by Oved Shisha

Received December 7, 1981

Klee [1] has shown that metric spaces containing discrete Chebyshev sets are subject to certain topological limitations. In this connection he posed the question whether a connected separable metric space can contain such a set.

We give an example, which answers this question in the affirmative. It is constructed as the union X of a sequence (S_n) of finite-dimensional simplices in l_2. They contain distinguished vertices b_n such that the distance of any point x in S_n from b_n is less than the distance between x and b_m for $m \neq n$. So the set $B = \{b_n : n = 0, 1, 2,...\}$ is a Chebyshev set in X. Moreover, by the given construction one achieves that B is $\frac{1}{2}$-dispersed and that, for each n, one vertex of S_n can be approximated by a sequence (x_m) with $x_m \in S_m$. This last fact guarantees the connectedness of X.

1

Let $e_{00}, e_{10}, e_{11},..., e_{n0}, e_{n1},..., e_{nn},...$ be an orthonormal set in l_2. We put

$$a_{00} = 0$$

and for $n = 1, 2,..., k = 0, 1,..., n-1$,

$$a_{nk} = \sum_{j=1}^{n-k-1} \frac{1}{2^{j+1}} e_{j0} + \frac{1}{2^{n+1}} \sum_{j=0}^{k} e_{nj}$$

(where we define $\sum_{j=l}^{k} a_j = 0$ if $l > k$).

* Present address: Department of Mathematics, University of Crete, Iraklion, Crete, Greece.

0021-9045/83 $3.00

For each $n = 2, 3, \ldots$, we have then

$$\|a_{n,n-1} - a_{n0}\| = \left\|\frac{1}{2^{n+1}} \sum_{j=1}^{n-1} e_{nj} - \sum_{j=1}^{n-1} \frac{1}{2^{j+1}} e_{j0}\right\|$$
$$\leqslant \frac{n-1}{2^{n+1}} + \sum_{j=1}^{n-1} \frac{1}{2^{j+1}} < \frac{1}{2^2} + \frac{1}{2} < 1.$$

Therefore we can put $\beta_n = (1 - \|\frac{1}{2}(a_{n,n-1} - a_{n0})\|^2)^{1/2}$ for $n = 1, 2, \ldots$, and consider the points

$$a_{01} = e_{00},$$
$$a_{nn} = \tfrac{1}{2}(a_{n,n-1} + a_{n0}) + \beta_n e_{nn}, \qquad n = 1, 2, \ldots.$$

LEMMA. *The above defined points a_{00}, a_{01}, a_{nk}, $k = 0, 1, \ldots, n$, $n = 1, 2, \ldots$, have the following properties:*

(i) $\lim_n a_{n,n-k-1} = a_{k0}$ *for each* $k = 0, 1, \ldots$.

(ii) *For each* $n = 3, 4, \ldots$, *the differences* $a_{n1} - a_{n0}$, $a_{n2} - a_{n1}, \ldots, a_{n,n-1} - a_{n,n-2}$ *are pairwise orthogonal.*

(iii) $\|a_{nn} - a_{nk}\| = 1$ *for all* $n \geqslant 1$, $k < n$.

(iv) $\|a_{nn} - a_{mk}\| > 1$ *for all* $n, m \geqslant 1$, $k < m$, $n \neq m$.

(v) $\|a_{nn} - a_{00}\| > 1$ *for all* $n \geqslant 1$.

(vi) $\|a_{01} - a_{mk}\| > 1$ *for all* $m \geqslant 1$, $k < m$.

(vii) $\|a_{nn} - a_{mm}\| > \frac{1}{2}$ *for all* $n, m \geqslant 1$, $n \neq m$.

(viii) $\|a_{01} - a_{nn}\| > \frac{1}{2}$ *for all* $n \geqslant 1$.

Proof. Properties (i) and (ii) are clear from the definitions.

(iii) By the definition of a_{nn} we have $a_{nn} - a_{nk} = \frac{1}{2}(a_{n,n-1} - a_{nk}) + \frac{1}{2}(a_{n0} - a_{nk}) + \beta_n e_{nn}$. By (ii) the differences $a_{n,n-1} - a_{nk}$ and $a_{n0} - a_{nk}$ are orthogonal to each other. Since they are also orthogonal to e_{nn}, we get

$$\|a_{nn} - a_{nk}\| = \|\tfrac{1}{2}(a_{n,n-1} - a_{nk}) - \tfrac{1}{2}(a_{n0} - a_{nk}) + \beta_n e_{nn}\|$$
$$= \|\tfrac{1}{2}(a_{n,n-1} - a_{n0}) + \beta_n e_{nn}\|.$$

It follows that

$$\|a_{nn} - a_{nk}\|^2 = \|\tfrac{1}{2}(a_{n,n-1} - a_{n0})\|^2 + \beta_n^2 = 1.$$

(iv) We observe that

$$\begin{aligned} a_{nn} - a_{n0} &= \tfrac{1}{2}(a_{n,n-1} - a_{n0}) + \beta_n e_{nn} \\ &= \frac{1}{2^{n+2}} \sum_{j=1}^{n-1} e_{nj} + \beta_n e_{nn} - \sum_{j=1}^{n-1} \frac{1}{2^{j+2}} e_{j0}, \end{aligned}$$

$$\begin{aligned} a_{nn} - a_{mk} &= \frac{1}{2^{n+2}} \sum_{j=1}^{n-1} e_{nj} + \beta_n e_{nn} - \frac{1}{2^{m+1}} \sum_{j=1}^{k} e_{mj} \\ &\quad + \sum_{j=1}^{n-1} \frac{1}{2^{j+2}} e_{j0} + \frac{1}{2^{n+1}} e_{n0} - \sum_{j=1}^{m+k-1} \frac{1}{2^{j+1}} e_{j0} - \frac{1}{2^{m+1}} e_{m0}. \end{aligned}$$

Comparing the coefficients in the above two formulas we see that $\|a_{nn} - a_{mk}\| > \|a_{nn} - a_{n0}\|$. The assertion follows then from (iii).

(v) Clearly we have

$$a_{nn} - a_{00} = a_{nn} = \frac{1}{2^{n+1}} e_{n0} + \frac{1}{2^{n+2}} \sum_{j=1}^{n-1} e_{nj} + \beta_n e_{nn} + \sum_{j=1}^{n-1} \frac{1}{2^{j+2}} e_{j0}.$$

Comparing with the first formula in the proof of (iv) we get $\|a_{nn} - a_{00}\| > \|a_{nn} - a_{n0}\| = 1$.

(vi) Obviously $\|a_{01} - a_{mk}\| = \|e_{00} - a_{mk}\| > \|e_{00}\| = 1$.

(vii) Assume that $n > m \geqslant 1$. Then we have

$$\begin{aligned} a_{nn} - a_{mm} &= \frac{1}{2^{n+2}} \sum_{j=1}^{n-1} e_{nj} + \beta_n e_{nn} - \frac{1}{2^{m+2}} \sum_{j=1}^{m-1} e_{mj} - \beta_m e_{mm} \\ &\quad - \frac{1}{2^{m+2}} e_{m0} + \sum_{j=m+1}^{n-1} \frac{1}{2^{j+2}} e_{j0} + \frac{1}{2^{n+1}} e_{n0}, \end{aligned}$$

therefore

$$\begin{aligned} \|a_{nn} - a_{mm}\| &> \left\| \frac{1}{2^{n+2}} \sum_{j=1}^{n-1} e_{nj} + \beta_n e_{nn} + \sum_{j=m+1}^{n-1} \frac{1}{2^{j+2}} e_{j0} + \frac{1}{2^{n+1}} e_{n0} \right\| \\ &= \left\| a_{nn} - \sum_{j=1}^{m} \frac{1}{2^{j+2}} e_{j0} \right\| \\ &\geqslant \|a_{nn}\| - \sum_{j=1}^{m} \frac{1}{2^{j+2}} > \|a_{nn}\| - \tfrac{1}{2}. \end{aligned}$$

Since $\|a_{nn}\| > 1$ (by (v)), we get $\|a_{nn} - a_{mm}\| > \frac{1}{2}$.

(viii) Obviously $\|a_{01} - a_{nn}\| = \|e_{00} - a_{nn}\| > \|e_{00}\| = 1$.

2

We are now able to construct the announced example. To this end we put

$$S_{00} = [a_{00}, a_{01}], \qquad S_n = co(\{a_{n0}, a_{n1}, \dots, a_{nn}\}) \qquad \text{for} \quad n = 1, 2, \dots.$$

The set $X = \bigcup_{n \geqslant 0} S_n$ is then a separable metric space, which is connected because of (i).

We consider now the points $b_0 = a_{01}$, $b_n = a_{nn}$ for $n = 1, 2, \dots$. By (vii) and (viii) the set $B = \{b_n : n = 0, 1, \dots\}$ is $\frac{1}{2}$-dispersed. Moreover, if x is a vertex of S_n and $n \neq m$, properties (iii)–(vi) imply that $\|x - b_n\| < \|x - b_m\|$. Since we consider the l_2-norm, the last inequality extends to all points of S_n proving that B is a Chebyshev set in X.

Reference

1. V. Klee, Dispersed Chebyshev sets and coverings by balls, *Math. Ann.* **257** (1981), 251–260.